

ON THE WEIGHTED L^2 ESTIMATE FOR THE k -CAUCHY-FUETER OPERATOR AND THE WEIGHTED k -BERGMAN KERNEL

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ABSTRACT. The k -Cauchy-Fueter operators, $k = 0, 1, \dots$, are quaternionic counterparts of the Cauchy-Riemann operator in the theory of several complex variables. The weighted L^2 method to solve Cauchy-Riemann equation is applied to find the canonical solution to the non-homogeneous k -Cauchy-Fueter equation in a weighted L^2 -space, by establishing the weighted L^2 estimate. The weighted k -Bergman space is the space of weighted L^2 integrable functions annihilated by the k -Cauchy-Fueter operator, as the counterpart of the Fock space of weighted L^2 -holomorphic functions on \mathbb{C}^n . We introduce the k -Bergman orthogonal projection to this closed subspace, which can be nicely expressed in terms of the canonical solution operator, and its matrix kernel function. We also find the asymptotic decay for this matrix kernel function.

1. INTRODUCTION

The k -Cauchy-Fueter operators over \mathbb{R}^{4n}

$$\mathcal{D}_0^{(k)} : C^\infty(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2) \longrightarrow C^\infty(\mathbb{R}^{4n}, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}),$$

$k = 0, 1, \dots$, are quaternionic counterparts of the Cauchy-Riemann operator $\bar{\partial}$ in the theory of several complex variables, where $\odot^p \mathbb{C}^2$ is the p -th symmetric tensor product of \mathbb{C}^2 . If we write a vector in the quaternionic space \mathbb{H}^n as $\mathbf{q} = (\mathbf{q}_0, \dots, \mathbf{q}_{n-1})$, the usual *Cauchy-Fueter operator* is defined as

$$\mathcal{D} : C^1(\mathbb{H}^n, \mathbb{H}) \rightarrow C(\mathbb{H}^n, \mathbb{H}^n), \quad \mathcal{D}f = \begin{pmatrix} \bar{\partial}_{\mathbf{q}_0} f \\ \vdots \\ \bar{\partial}_{\mathbf{q}_{n-1}} f \end{pmatrix},$$

for $f \in C^1(\mathbb{H}^n, \mathbb{H})$, where $\bar{\partial}_{\mathbf{q}_l} = \partial_{x_{4l+1}} + \mathbf{i}\partial_{x_{4l+2}} + \mathbf{j}\partial_{x_{4l+3}} + \mathbf{k}\partial_{x_{4l+4}}$, if we write $\mathbf{q}_l = x_{4l+1} + x_{4l+2}\mathbf{i} + x_{4l+3}\mathbf{j} + x_{4l+4}\mathbf{k} \in \mathbb{H}$, $l = 0, 1, \dots, n-1$. It is known that the Cauchy-Fueter operator coincides with the 1-Cauchy-Fueter operator [13]. In the quaternionic case, we have a family of operators acting on $\odot^k \mathbb{C}^2$ -valued functions, $k = 0, 1, \dots$, because $SU(2)$ as the group of unit quaternions has a family of irreducible representations $\odot^k \mathbb{C}^2$, while S^1 as the group of unit complex numbers has only one irreducible representation. The k -Cauchy-Fueter operators over \mathbb{R}^4 also have the origin in physics: they are the elliptic version of *spin $k/2$ massless field operators* over the Minkowski space (cf. e.g. [4] [11] [16] [17]): $\mathcal{D}_0^{(1)}\phi = 0$ corresponds to the Dirac-Weyl equation whose solutions correspond to neutrinos; $\mathcal{D}_0^{(2)}\phi = 0$ corresponds to the Maxwell equation whose solutions correspond to photons; $\mathcal{D}_0^{(3)}\phi = 0$ corresponds to the Rarita-Schwinger equation; $\mathcal{D}_0^{(4)}\phi = 0$ corresponds to linearized Einstein's equation whose solutions correspond to weak gravitational fields; etc..

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To develop the function theory of several quaternionic variables, we need to solve the *non-homogeneous k -Cauchy-Fueter equation*:

$$(1.1) \quad \mathcal{D}_0^{(k)} u = f,$$

where u is $\odot^k \mathbb{C}^2$ -valued and f is $\odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ -valued. Under the identification

$$(1.2) \quad \odot^k \mathbb{C}^2 \simeq \mathbb{C}^{k+1}, \quad \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n} \simeq \mathbb{C}^{2kn},$$

$\mathcal{D}_0^{(k)}$ is a $2kn \times (k+1)$ -matrix valued differential operator of the first order with constant coefficients. The equation (1.1) is overdetermined and its compatibility condition is that f is $\mathcal{D}_1^{(k)}$ -closed, i.e.

$$(1.3) \quad \mathcal{D}_1^{(k)} f = 0,$$

where $\mathcal{D}_1^{(k)}$ is the second operator in the k -Cauchy-Fueter complex:

$$(1.4) \quad 0 \rightarrow C^\infty(\mathbb{R}^{4n}, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0^{(k)}} C^\infty(\mathbb{R}^{4n}, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1^{(k)}} C^\infty(\mathbb{R}^{4n}, \mathcal{V}_2) \xrightarrow{\mathcal{D}_2^{(k)}} \dots,$$

and

$$(1.5) \quad \mathcal{V}_0 := \odot^k \mathbb{C}^2, \quad \mathcal{V}_1 := \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}, \quad \mathcal{V}_2 := \odot^{k-2} \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^{2n}.$$

Here $\wedge^2 \mathbb{C}^{2n}$ is the 2-th exterior product of \mathbb{C}^{2n} . These complexes play the role of Dolbeault complex in several complex variables, and are now explicitly known [21] (cf. also [1] [2] [3] [7] [8]).

The author [20] [21] solved the non-homogeneous k -Cauchy-Fueter equation in L^2 -space over \mathbb{R}^{4n} by using the method of classical harmonic analysis, and deduced Hartogs' phenomenon and integral representation formulae. In this paper, the weighted L^2 method to solve the $\bar{\partial}$ equation on \mathbb{C}^n (see e.g. [9] [12] [14] and references therein) is extended to solve the non-homogeneous k -Cauchy-Fueter equation (1.1). The L^2 method is a general method to deal with overdetermined systems of linear differential equations when we can establish the necessary L^2 estimate, e.g. it is applied to the Dirac operator in Clifford analysis [15]. The reason to consider the weighted L^2 -space is as follows. f is called *k -regular* if $\mathcal{D}_0^{(k)} f = 0$ in the sense of distributions. It is known that the space of k -regular polynomials are infinite dimensional (cf. [13]), and such functions are L^2 -integrable with Gaussian weight. This is similar to complex analysis, where one consider the space of L^2 -integrable holomorphic functions with Gaussian weight, called *Fock space*. Without a weight, a L^2 -integrable holomorphic (or k -regular) function must vanish. Given a nonnegative function φ , called a *weighted function*, consider the Hilbert space $L_\varphi^2(\mathbb{R}^{4n}, \mathbb{C})$ with the weighted inner product

$$(u, v)_\varphi := \int_{\mathbb{R}^{4n}} u \bar{v} e^{-2\varphi} dV,$$

where dV is the Lebesgue measure on \mathbb{R}^{4n} . For a complex linear space \mathcal{V} with an inner product $\langle \cdot, \cdot \rangle$ (e.g. $\mathcal{V} = \odot^k \mathbb{C}^2$ or $\odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$), we define $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V})$ with the weighted inner product

$$\langle f, g \rangle_\varphi := \int_{\mathbb{R}^{4n}} \langle f, g \rangle e^{-2\varphi} dV,$$

and the weighted norm $\|f\|_\varphi := \langle f, f \rangle_\varphi^{\frac{1}{2}}$. The *weighted k -Bergman space* with respect to weight $\varphi = |x|^2$ is then defined as

$$A_{(k)}^2(\mathbb{R}^{4n}, \varphi) := \left\{ f \in L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2); \mathcal{D}_0^{(k)} f = 0 \right\}.$$

It is infinite dimensional [13] because k -regular polynomials are integrable with respect to this weight.

In the sequel, we will drop the superscript for fixed k for simplicity.

$$(1.6) \quad L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_2)$$

is a complex, i.e. for any $u \in \text{Dom}(\mathcal{D}_0)$,

$$\mathcal{D}_0 u \in \text{Dom}(\mathcal{D}_1) \quad \text{and} \quad \mathcal{D}_1 \mathcal{D}_0 u = 0.$$

Then if $f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ is \mathcal{D}_1 -closed, the nonhomogeneous k -Cauchy-Fueter equation (1.1) has at most one solution $u \in \text{Dom}(\mathcal{D}_0)$ orthogonal to $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. If it exists, it is called the *canonical solution* to the nonhomogeneous k -Cauchy-Fueter equation (1.1). Consider the *associated Laplacian operator* $\square_\varphi : L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1) \rightarrow L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ given by

$$\square_\varphi := \mathcal{D}_0 \mathcal{D}_0^* + \mathcal{D}_1^* \mathcal{D}_1.$$

Theorem 1.1. *Suppose that $\varphi(x) = |x|^2$ and $k = 2, 3, \dots$. Then*

- (1) \square_φ has a bounded, self-adjoint and non-negative inverse N_φ such that

$$\|N_\varphi f\|_\varphi \leq \frac{1}{4} \|f\|_\varphi, \quad \text{for any } f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1).$$

- (2) $\mathcal{D}_0^* N_\varphi f$ is the canonical solution operator to the nonhomogeneous k -Cauchy-Fueter equation (1.1), i.e. if $f \in \text{Dom}(\mathcal{D}_1)$ is \mathcal{D}_1 -closed, then

$$\mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f$$

and $\mathcal{D}_0^* N_\varphi f$ orthogonal to $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. Moreover,

$$(1.7) \quad \|\mathcal{D}_0^* N_\varphi f\|_\varphi \leq \frac{1}{2} \|f\|_\varphi, \quad \|\mathcal{D}_1 N_\varphi f\|_\varphi \leq \frac{1}{2} \|f\|_\varphi.$$

The key step to prove this theorem is to establish the following weighted L^2 estimate.

Theorem 1.2. *Suppose that $\varphi(x) = |x|^2$ and $k = 2, 3, \dots$. Then*

$$(1.8) \quad 4 \|f\|_\varphi^2 \leq \|\mathcal{D}_0^* f\|_\varphi^2 + \|\mathcal{D}_1 f\|_\varphi^2$$

for any $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$.

The reason we only consider the weight $\varphi(x) = |x|^2$ is that the weighted L^2 estimate in this case is relatively easier. On \mathbb{R}^{4n} for $n > 1$, the operators $\mathcal{D}_0^{(0)}$ and $\mathcal{D}_1^{(1)}$ are differential operators of the second order, and the weighted L^2 estimate is more difficult in these cases. While on \mathbb{R}^4 , the k -Cauchy-Fueter complexes for $k = 0, 1$ are trivial. So we restrict to the case $k \geq 2$.

The weighted k -Bergman space $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ is a closed Hilbert subspace. We call the orthogonal projection $P : L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2) \rightarrow A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ the *weighted k -Bergman projection*. It can be nicely expressed in terms of the canonical solution operator as

$$(1.9) \quad Pf = f - \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f$$

for $f \in \text{Dom}(\mathcal{D}_0)$, as in the theory of several complex variables (cf. theorem 4.4.5 in [5]).

If we use the first isomorphism in (1.2), a function in $L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2)$ is \mathbb{C}^{k+1} -valued. The weighted k -Bergman projection P has a kernel $K(x, y)$ such that the following integral formula holds

$$(1.10) \quad f(x) = \int_{\mathbb{R}^{4n}} K(x, y) f(y) e^{-2\varphi} dV$$

for any $f \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. The kernel $K(x, y)$ is a $(k+1) \times (k+1)$ -matrix valued function, which is k -regular in variables x and anti- k -regular in variables y .

The main difference between the k -Cauchy-Fueter complexes and Dolbeault complex in the theory of several complex variables is that there exist symmetric forms except for the exterior forms. The analysis of exterior forms is classical, while the analysis of symmetric forms is relatively new. We can handle components of a $\odot^k \mathbb{C}^2$ - or $\odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ -valued function. Such notations are used by physicists as two-spinor notations for the massless field operators (cf. e.g. [16] [17] and references therein). They also appear in studying of quaternionic manifolds (cf. e.g. [22] and references therein).

The weighted L^2 estimate for the model case: $n = 1$ and $k = 2$, is obtain in section 2. The general case is proved in section 3. Based on the weighted L^2 estimate, Theorem 1.1 is proved in section 3. In section 4, we establish a localized a priori estimate for \square_φ and the Caccioppoli-type estimate, which hold for many systems of PDEs of the divergence form. From these estimates and the weighted L^2 estimate, we derive the asymptotic decay of the canonical solution $\mathcal{D}_0^* N_\varphi f$ to the nonhomogeneous k -Cauchy-Fueter equation (1.1) when f is compactly supported. Then by choosing suitable f in (1.9), we find the asymptotic estimate for the weighted k -Bergman kernel from the asymptotic behavior of the canonical solution.

Theorem 1.3. *Suppose that $\varphi(x) = |x|^2$ and $k = 2, 3, \dots$. Then we have the following pointwise estimate for the weighted k -Bergman kernel: there exists $\varepsilon > 0$ only depending on k, n such that*

$$(1.11) \quad |K(x, y)| \leq C e^{|x|^2 + |y|^2 + \frac{\varepsilon}{2}(|x| + |y|) - \varepsilon|x - y|}$$

for any $x, y \in \mathbb{R}^{4n}$ with $|x - y| > 3$, and some constant $C > 0$ only depending on k, n, ε .

The first estimate for the Bergman kernel of the weighted L^2 -holomorphic functions over the complex plane \mathbb{C} is due to Christ [6]. The result of Christ was extended by Delin [10] to several complex variables for strict plurisubharmonic weights. See also [9] [14] and references therein for recent results. Our estimate is a little bit weaker than the complex case because we have an extra factor $e^{\frac{\varepsilon}{2}(|x| + |y|)}$. But the estimate is the same when $|y|$ is larger compared to $|x|$ (cf. Remark 4.1).

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2. THE WEIGHTED L^2 ESTIMATE IN THE MODEL CASE: $n = 1$ AND $k = 2$

2.1. The complex vector fields $Z_{AA'}$'s on \mathbb{R}^{4n} and their formal adjoints. To give the definition of the k -Cauchy-Fueter operator, we need the following complex vector fields

$$(2.1) \quad (Z_{AA'}) := \begin{pmatrix} Z_{00'} & Z_{01'} \\ Z_{10'} & Z_{11'} \\ \vdots & \vdots \\ Z_{(2l)0'} & Z_{(2l)1'} \\ Z_{(2l+1)0'} & Z_{(2l+1)1'} \\ \vdots & \vdots \end{pmatrix} := \begin{pmatrix} \partial_{x_1} + \mathbf{i}\partial_{x_2} & -\partial_{x_3} - \mathbf{i}\partial_{x_4} \\ \partial_{x_3} - \mathbf{i}\partial_{x_4} & \partial_{x_1} - \mathbf{i}\partial_{x_2} \\ \vdots & \vdots \\ \partial_{x_{4l+1}} + \mathbf{i}\partial_{x_{4l+2}} & -\partial_{x_{4l+3}} - \mathbf{i}\partial_{x_{4l+4}} \\ \partial_{x_{4l+3}} - \mathbf{i}\partial_{x_{4l+4}} & \partial_{x_{4l+1}} - \mathbf{i}\partial_{x_{4l+2}} \\ \vdots & \vdots \end{pmatrix},$$

where $A = 0, \dots, 2n - 1$, $A' = 0', 1'$. This is motivated by the embedding of the quaternion algebra into the space of complex 2×2 -matrices:

$$x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \longmapsto \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}.$$

We will use

$$(2.2) \quad (\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to raise or lower primed indices, where $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$, i.e., $\sum_{B'=0',1'} \varepsilon_{A'B'} \varepsilon^{B'C'} = \delta_{A'}^{C'} = \sum_{B'=0',1'} \varepsilon^{C'B'} \varepsilon_{B'A'}$. For example,

$$Z_A^{A'} = \sum_{B'=0',1'} Z_{AB'} \varepsilon^{B'A'} = Z_{A0'} \varepsilon^{0'A'} + Z_{A1'} \varepsilon^{1'A'}.$$

In particular, we have $Z_A^{0'} = Z_{A1'}$, $Z_A^{1'} = -Z_{A0'}$ by

$$(2.3) \quad \varepsilon^{1'0'} = -\varepsilon^{0'1'} = 1, \quad \varepsilon^{0'0'} = \varepsilon^{1'1'} = 0$$

in (2.2). Then

$$(2.4) \quad (Z_A^{A'}) := \begin{pmatrix} Z_0^{0'} & Z_0^{1'} \\ Z_1^{0'} & Z_1^{1'} \\ \vdots & \vdots \\ Z_{(2n-2)}^{0'} & Z_{(2n-2)}^{1'} \\ Z_{(2n-1)}^{0'} & Z_{(2n-1)}^{1'} \end{pmatrix} := \begin{pmatrix} Z_{01'} & -Z_{00'} \\ Z_{11'} & -Z_{10'} \\ \vdots & \vdots \\ Z_{(2n-2)1'} & -Z_{(2n-2)0'} \\ Z_{(2n-1)1'} & -Z_{(2n-1)0'} \end{pmatrix}.$$

We also use

$$(2.5) \quad (\epsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ & \ddots \\ & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

and (ϵ^{AB}) , the inverse of (ϵ_{AB}) , to raise or lower unprimed indices, e.g. $Z_{A'}^A = \sum_{B=0}^{2n-1} Z_{A'B} \epsilon^{BA}$. The advantage of using raising indices is that the adjoint of $Z_A^{A'}$ can be written in a very simple form.

Proposition 2.1. (1) The formal adjoint operator Z_φ^* of a complex vector field Z is

$$Z_\varphi^* = -\overline{Z} + 2\overline{Z}\varphi.$$

(2) We have

$$(2.6) \quad Z^{AA'} = \overline{Z_{AA'}},$$

and the formal adjoint operator of $Z_A^{A'}$ is

$$(2.7) \quad (Z_A^{A'})_\varphi^* = Z_{A'}^A - 2Z_{A'}^A \varphi.$$

Proof. (1) For a complex vector field Z , we have

$$(Zu, v)_\varphi = (u, Z_\varphi^* v)_\varphi.$$

for $u, v \in C_0^\infty(\Omega, \mathbb{C})$. This is because

$$0 = \int_\Omega Z(u\overline{v}e^{-2\varphi})dV = \int_\Omega Zu \cdot \overline{v}e^{-2\varphi}dV + \int_\Omega u \cdot Z\overline{v} \cdot e^{-2\varphi}dV - 2 \int_\Omega u\overline{v} \cdot Z\varphi \cdot e^{-2\varphi}dV.$$

(2) By raising indices, $Z^{AA'} = \sum_{B=0}^{2n-1} \sum_{B'=0',1'} Z_{BB'} \epsilon^{BA} \epsilon^{B'A'}$. It is direct from definition of $Z_{AA'}$'s in (2.1) to see that

$$\begin{aligned} \overline{Z_{00'}} &= Z_{11'} = Z^{00'}, & \overline{Z_{10'}} &= -Z_{01'} = Z^{10'}, \\ \overline{Z_{01'}} &= -Z_{10'} = Z^{01'}, & \overline{Z_{11'}} &= Z_{00'} = Z^{11'}, \dots, \end{aligned}$$

by (2.3) and similar relations for ϵ^{AB} . Then $\overline{Z_{AA'}} = Z^{AA'}$. Since $(Z_A^{A'})^*_{\varphi} = -\overline{Z_A^{A'}} + 2\overline{Z_A^{A'}}\varphi$ by (1), and

$$(2.8) \quad \overline{Z_A^{A'}} = \sum_{B'=0',1'} \overline{Z_{AB'}} \epsilon^{B'A'} = - \sum_{B'=0',1'} Z^{AB'} \epsilon_{B'A'} = -Z_{A'}^A$$

we get (2.7). Here $\epsilon^{B'A'} = -\epsilon_{B'A'}$ by (2.2). \square

We will use the notations of the following complex differential operators:

$$(2.9) \quad \delta_{A'}^A := Z_{A'}^A - 2Z_{A'}^A \varphi,$$

for $A = 0, \dots, 2n-1$, $A' = 0', 1'$. Then we have $(Z_A^{A'})^*_{\varphi} = \delta_{A'}^A$, and

$$(2.10) \quad \left(Z_A^{A'} u, v \right)_{\varphi} = \left(u, \delta_{A'}^A v \right)_{\varphi}$$

for $u, v \in C_0^1(\Omega, \mathbb{C})$. By taking conjugate, we also have

$$(2.11) \quad \left(\delta_{A'}^A u, v \right)_{\varphi} = \left(u, Z_A^{A'} v \right)_{\varphi}.$$

2.2. The weighted L^2 estimate in the model case $n = 1$ and $k = 2$. In this case,

$$(2.12) \quad \mathcal{V}_0 := \odot^2 \mathbb{C}^2 \cong \mathbb{C}^3, \quad \mathcal{V}_1 := \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4, \quad \mathcal{V}_2 := \wedge^2 \mathbb{C}^2 \cong \mathbb{C}^1.$$

By definition, $\odot^2 \mathbb{C}^2$ is a subspace of $\otimes^2 \mathbb{C}^2$, and an element f of $L_{\varphi}^2(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$ has 4 components $f_{0'0'}, f_{1'0'}, f_{0'1'}$ and $f_{1'1'}$ such that $f_{1'0'} = f_{0'1'}$. Its L^2 inner product is induced from that of $L_{\varphi}^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$ by

$$\langle f, g \rangle_{\varphi} = \sum_{A', B'=0', 1'} (f_{A'B'}, g_{A'B'})_{\varphi} = (f_{0'0'}, g_{0'0'})_{\varphi} + 2(f_{0'1'}, g_{0'1'})_{\varphi} + (f_{1'1'}, g_{1'1'})_{\varphi};$$

$f \in L_{\varphi}^2(\mathbb{R}^4, \mathbb{C}^2 \otimes \mathbb{C}^2)$ has 4 components $f_{A'A}$, $A = 0, 1$, $A' = 0', 1'$, and

$$\langle f, g \rangle_{\varphi} = \sum_{A=0,1} \sum_{A'=0',1'} (f_{A'A}, g_{A'A})_{\varphi};$$

while $f \in L_{\varphi}^2(\mathbb{R}^4, \wedge^2 \mathbb{C}^2)$ has components f_{AB} with $f_{AB} = -f_{BA}$, among which there is only one nontrivial (i.e. $f_{00} = f_{11} = 0$, $f_{01} = -f_{10}$), and

$$\langle f, g \rangle_{\varphi} = \sum_{A,B=0,1} (f_{AB}, g_{AB})_{\varphi} = 2(f_{01}, g_{01})_{\varphi}.$$

The operators in the 2-Cauchy-Fueter complex over \mathbb{R}^4 are given by

$$(2.13) \quad (\mathcal{D}_0 \phi)_{A'A} := \sum_{B'=0',1'} Z_A^{B'} \phi_{B'A'} = Z_A^{0'} \phi_{0'A'} + Z_A^{1'} \phi_{1'A'},$$

for $\phi \in C^1(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$ where $A = 0, 1$, $A' = 0', 1'$, and

$$(2.14) \quad (\mathcal{D}_1 \psi)_{AB} := 2 \sum_{A'=0',1'} Z_{[A}^{A'} \psi_{B]A'} = \sum_{A'=0',1'} (Z_A^{A'} \psi_{BA'} - Z_B^{A'} \psi_{AA'})$$

for $\psi \in C^1(\mathbb{R}^4, \mathbb{C}^2 \otimes \mathbb{C}^2)$, where

$$h_{[AB]} := \frac{1}{2}(h_{AB} - h_{BA})$$

is the antisymmetrisation. Here and in the sequel, we write $\psi_{AA'} := \psi_{A'A}$ for convenience. It is direct to see that

$$\begin{aligned} (\mathcal{D}_1 \mathcal{D}_0 \phi)_{AB} &= \sum_{A'=0',1'} \left(Z_A^{A'} (\mathcal{D}_0 \phi)_{BA'} - Z_B^{A'} (\mathcal{D}_0 \phi)_{AA'} \right) \\ (2.15) \quad &= \sum_{A', C'=0',1'} \left(Z_A^{A'} Z_B^{C'} \phi_{C'A'} - Z_B^{A'} Z_A^{C'} \phi_{C'A'} \right) = 0 \end{aligned}$$

by relabeling indices, $\phi_{C'A'} = \phi_{A'C'}$ and the commutativity $\nabla_B^{A'} \nabla_A^{C'} = \nabla_A^{C'} \nabla_B^{A'}$, as scalar differential operators of constant complex coefficients (cf. (2.11) in [4]).

Lemma 2.1. (1) For any $h \in L_\varphi^2(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$ and $H \in L_\varphi^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$(2.16) \quad \sum_{A', B'} (h_{A'B'}, H_{A'B'})_\varphi = \sum_{A', B'} (h_{A'B'}, H_{(A'B')})_\varphi,$$

where

$$H_{(A'B')} := \frac{1}{2}(H_{A'B'} + H_{B'A'})$$

is the symmetrisation, i.e. $(H_{(A'B')}) \in L_\varphi^2(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$.

(2) For any $h \in L_\varphi^2(\mathbb{R}^4, \wedge^2 \mathbb{C}^2)$ and $H \in L_\varphi^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$(2.17) \quad \sum_{A, B} (h_{AB}, H_{AB})_\varphi = \sum_{A, B} (h_{AB}, H_{[AB]})_\varphi.$$

(3) For any $h, H \in L_\varphi^2(\mathbb{R}^4, \otimes^2 \mathbb{C}^2)$, we have

$$(2.18) \quad \sum_{A, B} (h_{BA}, H_{AB})_\varphi = \sum_{A, B} (h_{AB}, H_{AB})_\varphi - 2 \sum_{A, B} (h_{[AB]}, H_{[AB]})_\varphi.$$

Proof. (1) This is because

$$\sum_{A', B'} h_{A'B'} \overline{H_{(A'B')}} = \frac{1}{2} \sum_{A', B'} h_{A'B'} (\overline{H_{A'B'}} + \overline{H_{B'A'}}) = \sum_{A', B'} h_{A'B'} \overline{H_{A'B'}}$$

by changing indices and $h_{A'B'} = h_{B'A'}$.

(2) This is because

$$(2.19) \quad \sum_{A, B} h_{AB} \overline{H_{AB}} = \frac{1}{2} \sum_{A, B} h_{AB} (\overline{H_{AB}} - \overline{H_{BA}}) = \sum_{A, B} h_{AB} \overline{H_{[AB]}}$$

by changing indices and $h_{BA} = -h_{AB}$.

(3) This is because

$$(2.20) \quad \sum_{A, B} h_{BA} \overline{H_{AB}} = \sum_{A, B} h_{AB} \overline{H_{AB}} + \sum_{A, B} (h_{BA} - h_{AB}) \overline{H_{AB}}$$

and the second term in the right hand side is $-2 \sum_{A, B} h_{[AB]} \overline{H_{AB}} = -2 \sum_{A, B} h_{[AB]} \overline{H_{[AB]}}$ by the identity (2.19). \square

Lemma 2.2. For $f \in C_0^\infty(\mathbb{R}^4, \mathbb{C}^2 \otimes \mathbb{C}^2)$, we have

$$(2.21) \quad (\mathcal{D}_0^* f)_{A'B'} = \sum_{A=0,1} \delta_{(A')A}^A f_{B'A}.$$

Proof. For any $g \in C_0^\infty(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$, we have

$$\begin{aligned} \langle \mathcal{D}_0 g, f \rangle_\varphi &= \sum_{A,A',B'} \left(Z_A^{A'} g_{A'B'}, f_{B'A} \right)_\varphi = \sum_{A,A',B'} (g_{A'B'}, \delta_{(A')A}^A f_{B'A})_\varphi \\ &= \sum_{A',B'} \left(g_{A'B'}, \sum_A \delta_{(A')A}^A f_{B'A} \right)_\varphi = \langle g, \mathcal{D}_0^* f \rangle_\varphi \end{aligned}$$

by using (2.10) and Lemma 2.1 (1). Here we have to symmetrise $(A'B')$ in $\sum_A \delta_{(A')A}^A f_{B'A}$ since only after symmetrisation it becomes an element of $C_0^\infty(\mathbb{R}^4, \odot^2 \mathbb{C}^2)$, i.e. a $\odot^2 \mathbb{C}^2$ -valued function. \square

Theorem 2.1. Suppose that there exist a constant $c > 0$ such that the weight φ satisfies

$$(2.22) \quad \sum_{A,B,A',B'} Z_B^{A'} \overline{Z_A^{B'}} \varphi(x) \cdot \xi_{A'A} \overline{\xi_{B'B}} \geq c \sum_{A,A'} |\xi_{A'A}|^2.$$

for any $x \in \mathbb{R}^{4n}$ and $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$. Then we have the weighted L^2 estimate

$$(2.23) \quad c \|f\|_\varphi^2 \leq \|\mathcal{D}_0^* f\|_\varphi^2 + \|\mathcal{D}_1 f\|_\varphi^2,$$

for any $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$.

Proof. By definition, we have $\text{Dom}(\mathcal{D}_1) := \{f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1); \mathcal{D}_1 f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_2)\}$. Then \mathcal{D}_1 is densely-defined since $C_0^\infty(\mathbb{R}^{4n}, \mathcal{V}_1)$ is contained in its domain. It is also closed since differentiation is continuous on distributions. So is \mathcal{D}_0^* as a differential operator given by (2.21). Therefore it is sufficient to show (2.23) for $f \in C_0^\infty(\mathbb{R}^{4n}, \mathcal{V}_1)$. It follows from the definition of \mathcal{D}_0 in (2.13), \mathcal{D}_0^* in Lemma 2.2 and the definition of symmetrisation that

$$\begin{aligned} 2\langle \mathcal{D}_0^* f, \mathcal{D}_0^* f \rangle_\varphi &= 2\langle \mathcal{D}_0 \mathcal{D}_0^* f, f \rangle_\varphi = 2 \sum_{B,B'} \left(\sum_{A'} Z_B^{A'} \sum_A \delta_{(A')A}^A f_{B'A}, f_{B'B} \right)_\varphi \\ (2.24) \quad &= \sum_{A,B,A',B'} \left(Z_B^{A'} \delta_{A'A}^A f_{B'A}, f_{B'B} \right)_\varphi + \sum_{A,B,A',B'} \left(Z_B^{A'} \delta_{B'B}^A f_{A'A}, f_{B'B} \right)_\varphi := \Sigma_0 + \Sigma_1, \end{aligned}$$

where

$$(2.25) \quad \Sigma_0 = \sum_{A',B'} \left(\sum_A \delta_{A'A}^A f_{B'A}, \sum_B \delta_{B'B}^B f_{B'B} \right)_\varphi = \sum_{A',B'} \left\| \sum_A \delta_{A'A}^A f_{B'A} \right\|_\varphi^2 \geq 0,$$

and

$$\begin{aligned} (2.26) \quad \Sigma_1 &= \sum_{A,B,A',B'} \left\{ \left(\delta_{B'B}^A Z_B^{A'} f_{AA'}, f_{BB'} \right)_\varphi + \left([Z_B^{A'}, \delta_{B'B}^A] f_{AA'}, f_{BB'} \right)_\varphi \right\} \\ &= \sum_{A,B,A',B'} \left\{ \left(Z_B^{A'} f_{AA'}, Z_B^{B'} f_{BB'} \right)_\varphi + 2 \left(Z_B^{A'} \overline{Z_A^{B'}} \varphi \cdot f_{AA'}, f_{BB'} \right)_\varphi \right\}, \end{aligned}$$

by using the formal adjoint operator (2.11), relabeling indices and using the commutator

$$(2.27) \quad \left[Z_B^{A'}, \delta_{B'B}^A \right] = -2Z_B^{A'} Z_{B'}^A \varphi = 2Z_B^{A'} \overline{Z_A^{B'}} \varphi,$$

which follows from (2.8)-(2.9) and the commutativity $Z_B^{A'} Z_B^A = Z_B^A Z_B^{A'}$ as scalar differential operators of constant coefficients. The first summation in the right hand side of (2.26) is equal to

$$\begin{aligned} \sum_{A,B,A',B'} \left(Z_B^{A'} f_{AA'}, Z_A^{B'} f_{BB'} \right)_\varphi &= \sum_{A,B} \left(\sum_{A'} Z_B^{A'} f_{AA'}, \sum_{B'} Z_A^{B'} f_{BB'} \right)_\varphi \\ &= \sum_{A,B} \left\| \sum_{A'} Z_A^{A'} f_{BA'} \right\|_\varphi^2 - 2 \sum_{A,B} \left\| \sum_{A'} Z_{[A}^{A'} f_{B]A'} \right\|_\varphi^2 \\ &= \sum_{A,B} \left\| \sum_{A'} Z_A^{A'} f_{BA'} \right\|_\varphi^2 - \frac{1}{2} \|\mathcal{D}_1 f\|_\varphi^2 \end{aligned}$$

by applying (2.18) with $h_{BA} = \sum_{A'} Z_B^{A'} f_{AA'}$, $H_{AB} = \sum_{B'} Z_A^{B'} f_{BB'}$. Now substituting (2.25)-(2.26) into (2.24) and using the above identity, we get

$$\begin{aligned} 2 \|\mathcal{D}_0^* f\|_\varphi^2 + \frac{1}{2} \|\mathcal{D}_1 f\|_\varphi^2 &= 2 \sum_{A,B,A',B'} \left(Z_B^{A'} \overline{Z_A^{B'}} \varphi \cdot f_{AA'}, f_{BB'} \right)_\varphi \\ (2.28) \quad &+ \sum_{A',B'} \left\| \sum_A \delta_{A'}^A f_{B'A} \right\|_\varphi^2 + \sum_{A,B} \left\| \sum_{A'} Z_A^{A'} f_{BA'} \right\|_\varphi^2. \end{aligned}$$

Now the resulting estimate follows from the assumption (2.22) for φ . \square

Remark 2.1. (1) We do not handle the term Σ_0 in (2.25) by using commutators. Because if we do so

$$\begin{aligned} \Sigma_0 &= \sum_{A,B,A',B'} (\delta_{A'}^A Z_B^{A'} f_{AB'}, f_{BB'}) + ([Z_B^{A'}, \delta_{A'}^A] f_{AB'}, f_{BB'}) \\ &= \sum_{A,B,A',B'} \left(Z_B^{A'} f_{AB'}, Z_A^{A'} f_{BB'} \right)_\varphi + 2(Z_B^{A'} \overline{Z_A^{A'}} \varphi f_{AB'}, f_{BB'}), \end{aligned}$$

the first term in the right hand side above is quite difficult to control. But over \mathbb{R}^4 it can be controlled in terms of $\mathcal{D}_0^* f$ and $\mathcal{D}_1 f$. Based on such estimates, we can solve the Neumann problem for the k -Cauchy-Fueter complexes over k -pseudoconvex domains in \mathbb{R}^4 (cf. [23]).

(2) $\varphi = |x|^2$ satisfies the assumption (2.22) for φ with $c = 4$ by the following Lemma 3.2.

3. THE CANONICAL SOLUTION OPERATOR TO THE NONHOMOGENEOUS k -CAUCHY-FUETER EQUATION

3.1. The weighted L^2 estimate in the general case. Recall that the symmetric power $\odot^k \mathbb{C}^2$ is a subspace of $\otimes^k \mathbb{C}^2$, and an element of $\odot^k \mathbb{C}^2$ is given by a 2^k -tuple $(f_{A'_1 \dots A'_k}) \in \otimes^k \mathbb{C}^2$ with $A'_1 \dots A'_k = 0', 1'$, where $f_{A'_1 \dots A'_k}$ is invariant under permutations of subscripts, i.e.

$$f_{A'_1 \dots A'_k} = f_{A'_{\sigma(1)} \dots A'_{\sigma(k)}},$$

for any $\sigma \in S_k$, the group of permutations of k letters. Note that $\dim(\odot^k \mathbb{C}^2) = k + 1$ (cf. (4.1)) while $\dim(\otimes^k \mathbb{C}^2) = 2^k$. An element of the exterior power $\wedge^2 \mathbb{C}^{2n}$ is given by a tuple (f_{AB}) with $f_{AB} = -f_{BA}$, $A, B = 0, \dots, 2n-1$. An element of $\odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ is given by a tuple $(f_{A'_2 \dots A'_k A}) \in \otimes^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n}$, which is invariant under permutations of A'_2, \dots, A'_k . We will use symmetrisation of primed indices

$$(3.1) \quad f_{\dots(A'_1 \dots A'_k) \dots} := \frac{1}{k!} \sum_{\sigma \in S_k} f_{\dots A'_{\sigma(1)} \dots A'_{\sigma(k)} \dots}$$

The first two operators in k -Cauchy-Fueter complex (1.4)-(1.5) over \mathbb{R}^{4n} are given by

$$(3.2) \quad \begin{aligned} (\mathcal{D}_0 f)_{A'_2 \dots A'_k A} &:= \sum_{A'_1=0', 1'} Z_A^{A'_1} f_{A'_1 A'_2 \dots A'_k} = Z_A^{0'} f_{0' A'_2 \dots A'_k} + Z_A^{1'} f_{1' A'_2 \dots A'_k}, \\ (\mathcal{D}_1 h)_{ABA'_3 \dots A'_k} &:= 2 \sum_{A'=0', 1'} Z_{[A}^{A'} h_{B]A' A'_3 \dots A'_k} = \sum_{A'=0', 1'} \left(Z_A^{A'} h_{BA' A'_3 \dots A'_k} - Z_B^{A'} h_{AA' A'_3 \dots A'_k} \right), \end{aligned}$$

for $f \in C^1(\mathbb{R}^{4n}, \mathcal{V}_0)$, $h \in C^1(\mathbb{R}^{4n}, \mathcal{V}_1)$, where $A, B = 0, 1, \dots, 2n-1$, $A'_2, \dots, A'_k = 0', 1'$. Here and in the sequel, we write $h_{AA'_2 A'_3 \dots A'_k} := h_{A'_2 A'_3 \dots A'_k A}$ for convenience. It is direct to check that $\mathcal{D}_1 \circ \mathcal{D}_0 = 0$ as (2.15).

The weighted inner product of $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0)$ is induced from that of $L_\varphi^2(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2)$. Namely we define

$$\langle f, h \rangle_\varphi := \sum_{A'_1, \dots, A'_k} \left(f_{A'_1 \dots A'_k}, h_{A'_1 \dots A'_k} \right)_\varphi$$

for $f, h \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0)$, and $\|f\|_\varphi = \langle f, f \rangle_\varphi^{\frac{1}{2}}$. We define the weighted induced inner products of $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ and $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_2)$ similarly.

Lemma 3.1. *For $f \in C_0^\infty(\mathbb{R}^{4n}, \mathcal{V}_1)$, we have*

$$(3.3) \quad (\mathcal{D}_0^* f)_{A'_1 A'_2 \dots A'_k} = \sum_{A=0}^{2n-1} \delta_{(A'_1}^A f_{A'_2 \dots A'_k)A}.$$

Proof. For any $g \in C_0^\infty(\mathbb{R}^{4n}, \mathcal{V}_0)$ we have

$$\begin{aligned} \langle \mathcal{D}_0 g, f \rangle_\varphi &= \sum_{A, A'_2, \dots, A'_k} \left(\sum_{A'_1} Z_A^{A'_1} g_{A'_1 \dots A'_k}, f_{A'_2 \dots A'_k A} \right)_\varphi = \sum_{A, A'_1, \dots, A'_k} \left(g_{A' A'_1 \dots A'_k}, \delta_{A'_1}^A f_{A'_2 \dots A'_k A} \right)_\varphi \\ &= \sum_{A'_1, \dots, A'_k} \left(g_{A'_1 \dots A'_k}, \sum_A \delta_{(A'_1}^A f_{A'_2 \dots A'_k)A} \right)_\varphi = \langle g, \mathcal{D}_0^* f \rangle_\varphi \end{aligned}$$

by using (2.10) and symmetrisation

$$(3.4) \quad \sum_{A'_1, \dots, A'_k} \left(g_{A'_1 \dots A'_k}, G_{A'_1 \dots A'_k} \right)_\varphi = \sum_{A'_1, \dots, A'_k} \left(g_{A'_1 \dots A'_k}, G_{(A'_1 \dots A'_k)} \right)_\varphi$$

for any $g \in L_\varphi^2(\mathbb{R}^{4n}, \odot^k \mathbb{C}^2)$, $G \in L_\varphi^2(\mathbb{R}^{4n}, \otimes^k \mathbb{C}^2)$. Here we have to symmetrise indices $(A'_1 \dots A'_k)$ in $\sum_A \delta_{A'_1}^A f_{A'_2 \dots A'_k A}$ since only after symmetrisation it becomes an element of $C_0^\infty(\mathbb{R}^{4n}, \mathcal{V}_0)$, i.e. a $\odot^k \mathbb{C}^2$ -valued function. (3.4) is a generalization of Lemma 2.1 (1). It holds because

$$R.H.S. = \frac{1}{k!} \sum_{A'_1, \dots, A'_p} \sum_{\sigma \in S_k} \left(g_{A'_1 \dots A'_k}, G_{A'_{\sigma(1)} \dots A'_{\sigma(k)}} \right)_\varphi = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{A'_1, \dots, A'_k} \left(g_{A'_{\sigma^{-1}(1)} \dots A'_{\sigma^{-1}(k)}}, G_{A'_1 \dots A'_k} \right)_\varphi$$

by relabeling indices, which equals to L.H.S. by g symmetric in the indices, i.e. $g_{A'_{\sigma^{-1}(1)} \dots A'_{\sigma^{-1}(k)}} = g_{A'_1 \dots A'_k}$ for any permutation σ . \square

Proof of Theorem 1.2. As in the model case $n = 1$, $k = 2$, it is sufficient to show the weighted L^2 -estimate (1.8) for $f \in C_0^\infty(\mathbb{R}^{4n}, \odot^{k-1} \mathbb{C}^2 \otimes \mathbb{C}^{2n})$. Recall that if $(F_{A'_1 \dots A'_k}) \in \otimes^k \mathbb{C}^2$ is symmetric in $A'_2 \dots A'_k$, then we have

$$(3.5) \quad F_{(A'_1 \dots A'_k)} = \frac{1}{k!} (F_{A'_1 A'_2 \dots A'_k} + \dots + F_{A'_s A'_2 \dots A'_1 \dots A'_k} + \dots + F_{A'_k A'_2 \dots A'_1}),$$

by definition of symmetrisation (3.1). Now we expand the symmetrisation to get

$$\begin{aligned}
k\langle \mathcal{D}_0^* f, \mathcal{D}_0^* f \rangle_\varphi &= k\langle \mathcal{D}_0 \mathcal{D}_0^* f, f \rangle_\varphi \\
&= k \sum_{B, A'_2, \dots, A'_k} \left(\sum_{A'_1} Z_B^{A'_1} \sum_A \delta_{(A'_1 f_{A'_2 \dots A'_k} A), f_{A'_2 \dots A'_k} B} \right)_\varphi \\
&= \sum_{A, B, A'_1, \dots, A'_k} \left(Z_B^{A'_1} \delta_{A'_1}^A f_{A'_2 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi + \sum_{A, B, A'_1, \dots, A'_k} \sum_{s=2}^k \left(Z_B^{A'_1} \delta_{A'_s}^A f_{\dots A'_1 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi \\
&=: \Sigma_0 + \Sigma_1,
\end{aligned}$$

by the adjoint operator \mathcal{D}_0^* in Lemma 3.1. Here we split the sum into the cases $s = 1$ and $s \geq 2$ as in the model case (cf. Remark 2.1). Note that

$$\Sigma_0 = \sum_{A'_1, \dots, A'_k} \left(\sum_A \delta_{A'_1}^A f_{A'_2 \dots A'_k} A, \sum_B \delta_{A'_1}^B f_{A'_2 \dots A'_k} B \right)_\varphi \geq 0$$

by using (2.10), and

$$\begin{aligned}
\Sigma_1 &= \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} \left(\delta_{A'_s}^A Z_B^{A'_1} f_{A'_2 \dots A'_1 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi + \left(\left[Z_B^{A'_1}, \delta_{A'_s}^A \right] f_{A'_2 \dots A'_1 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi \\
&= \Sigma'_2 + \Sigma''_2
\end{aligned}$$

by using commutators. For the second sum,

$$\begin{aligned}
\Sigma''_2 &= 2 \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} \left(Z_B^{A'_1} \overline{Z_A^{A'_s}} \varphi \cdot f_{A'_2 \dots A'_1 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi \\
&= 8 \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} \left(\delta_{BA} \delta_{A'_1 A'_s} \cdot f_{A'_2 \dots A'_1 \dots A'_k} A, f_{A'_2 \dots A'_k} B \right)_\varphi = 8(k-1) \|f\|_\varphi^2
\end{aligned}$$

for $\varphi(x) = |x|^2$ by the following Lemma 3.2 and f symmetric in the primed indices. On the other hand,

$$\begin{aligned}
\Sigma'_2 &= \sum_{s=2}^k \sum_{A, B, A'_1, \dots, A'_k} \left(Z_B^{A'_1} f_{A'_2 \dots A'_1 \dots A'_k} A, Z_A^{A'_s} f_{A'_2 \dots A'_k} B \right)_\varphi \\
&= \sum_{s=2}^k \sum_{A, B} \sum_{\widehat{A'_1}, \dots, \widehat{A'_s}, \dots, A'_k} \left(\sum_{A'_1} Z_B^{A'_1} f_{A'_1 \dots \widehat{A'_s} \dots A'_k} A, \sum_{A'_s} Z_A^{A'_s} f_{A'_s A'_2 \dots \widehat{A'_s} \dots A'_k} B \right)_\varphi \\
&= (k-1) \sum_{B'_3, \dots, B'_k=0', 1'} \sum_{A, B} \left(\sum_{A'} Z_B^{A'} f_{A' B'_3 \dots B'_k} A, \sum_{A'} Z_A^{A'} f_{A' B'_3 \dots B'_k} B \right)_\varphi
\end{aligned}$$

by f symmetric in the primed indices and relabelling indices. Then applying Lemma 2.1 (3) ((2.20) holds for $A, B = 0, \dots, 2n-1$) to the right hand side with $h_{BA} = \sum_{A'} Z_B^{A'} f_{AA' B'_3 \dots B'_k}$ and $H_{AB} =$

$\sum_{A'} Z_A^{A'} f_{BA'B'_3 \dots B'_k}$ for fixed B'_3, \dots, B'_k , we get

$$\begin{aligned} \Sigma'_2 &= (k-1) \sum_{B'_3, \dots, B'_k} \sum_{A, B} \left\{ \left\| \sum_{A'} Z_A^{A'} f_{BA'B'_3 \dots B'_k} \right\|_\varphi^2 - 2 \left\| \sum_{A'} Z_{[A}^{A'} f_{B]A'B'_3 \dots B'_k} \right\|_\varphi^2 \right\} \\ &= (k-1) \sum_{A, B, B'_3, \dots, B'_k} \left\| \sum_{A'} Z_A^{A'} f_{BA'B'_3 \dots B'_k} \right\|_\varphi^2 - \frac{k-1}{2} \|\mathcal{D}_1 f\|_\varphi^2. \end{aligned}$$

Now we get

$$k \|\mathcal{D}_0^* f\|_\varphi^2 + \frac{k-1}{2} \|\mathcal{D}_1 f\|_\varphi^2 \geq 8(k-1) \|f\|_\varphi^2.$$

The estimate (1.8) follows. \square

Lemma 3.2. $Z_B^{A'} \overline{Z_A^{B'}} |x|^2 = 4\delta_{AB} \delta_{A'B'}$. In particular, $\varphi = |x|^2$ satisfies the assumption (2.22) for φ with $c = 4$.

To prove this lemma, we introduce complex linear functions

$$(3.6) \quad (z_{AA'}) := \begin{pmatrix} z_{00'} & z_{01'} \\ z_{10'} & z_{11'} \\ \vdots & \vdots \\ z_{(2l)0'} & z_{(2l)1'} \\ z_{(2l+1)0'} & z_{(2l+1)1'} \\ \vdots & \vdots \end{pmatrix} := \begin{pmatrix} x_1 - \mathbf{i}x_2 & -x_3 + \mathbf{i}x_4 \\ x_3 + \mathbf{i}x_4 & x_1 + \mathbf{i}x_2 \\ \vdots & \vdots \\ x_{4l+1} - \mathbf{i}x_{4l+2} & -x_{4l+3} + \mathbf{i}x_{4l+4} \\ x_{4l+3} + \mathbf{i}x_{4l+4} & x_{4l+1} + \mathbf{i}x_{4l+2} \\ \vdots & \vdots \end{pmatrix},$$

where $A = 0, \dots, 2n-1$, $A' = 0', 1'$. $z_{AA'}$ is obtained by replacing ∂_{x_j} in $\overline{Z_{AA'}}$ in (2.1) by x_j . By the following lemma, $z_{AA'}$'s can be viewed as independent variables and $Z_{AA'}$'s are derivatives with respect to these variables formally.

Lemma 3.3. $Z_{AA'} z_{BB'} = 2\delta_{AB} \delta_{A'B'}$.

Proof. Assume that $A = 2l, A' = 0'$. By (3.6), we have

$$\begin{aligned} Z_{(2l)0'} z_{(2l)0'} &= (\partial_{x_{4l+1}} + \mathbf{i}\partial_{x_{4l+2}})(x_{4l+1} - \mathbf{i}x_{4l+2}) = 2; \\ Z_{(2l)0'} z_{(2l+1)1'} &= (\partial_{x_{4l+1}} + \mathbf{i}\partial_{x_{4l+2}})(x_{4l+1} + \mathbf{i}x_{4l+2}) = 0. \end{aligned}$$

Note that $Z_{(2l)0'}$ is a differential operator with respect to variables x_{4l+1} and x_{4l+2} , while $z_{BB'}$ for $BB' \neq (2l)0'$ or $(2l+1)1'$ is independent of variables x_{4l+1} and x_{4l+2} . So we get

$$Z_{(2l)0'} z_{BB'} = 0$$

for such BB' . It is similar to check the result directly for other vectors $Z_{(2l)1'}$, $Z_{(2l+1)0'}$ and $Z_{(2l+1)1'}$. \square

Proof of Lemma 3.2. Note that $(\partial_{x_j} \pm \mathbf{i}\partial_{x_k})|x|^2 = 2(x_j \pm \mathbf{i}x_k)$. So $\overline{Z_{AC'}}|x|^2 = 2z_{AC'}$ by definitions of $\overline{Z_{AC'}}$'s and $z_{AC'}$'s in (3.6). Then we have

$$\begin{aligned} Z_B^{A'} \overline{Z_A^{B'}} |x|^2 &= \sum_{D', C'} Z_{BD'} \overline{Z_{AC'}} |x|^2 \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} = 2 \sum_{D', C'} Z_{BD'} z_{AC'} \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} \\ &= 4 \sum_{D', C'} \delta_{AB} \delta_{C'D'} \cdot \varepsilon^{C'B'} \varepsilon^{D'A'} = 4\delta_{AB} \delta_{A'B'}, \end{aligned}$$

by Lemma 3.3. Here by (2.3), $\varepsilon^{C'B'}\varepsilon^{C'A'} = 1$ only if $A' = B'$ and C' is different from them. Otherwise, it vanishes. So for any $(\xi_{A'A}) \in \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\sum_{A,B,A',B'} Z_B^{A'} \overline{Z_A^{B'}} |x|^2 \cdot \xi_{A'A} \overline{\xi_{B'B}} = 4|\xi|^2.$$

3.2. The associated Laplacian operator \square_φ . By definition,

$$\text{Dom}(\square_\varphi) := \{f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1); f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1), \mathcal{D}_0^* f \in \text{Dom}(\mathcal{D}_0), \mathcal{D}_1 f \in \text{Dom}(\mathcal{D}_1^*)\}.$$

We introduce

$$\mathcal{E}_\varphi(f, g) := \langle \mathcal{D}_0^* f, \mathcal{D}_0^* g \rangle_\varphi + \langle \mathcal{D}_1 f, \mathcal{D}_1 g \rangle_\varphi$$

for any $f, g \in \text{Dom}(\mathcal{E}_\varphi) := \text{Dom}(\mathcal{D}_1) \cap \text{Dom}(\mathcal{D}_0^*)$. By definition of adjoint operators, we have

$$(3.7) \quad \mathcal{E}_\varphi(f, g) = \langle \square_\varphi f, g \rangle_\varphi$$

for any $f \in \text{Dom}(\square_\varphi), g \in \text{Dom}(\mathcal{E}_\varphi)$.

Note that for any $F \in \text{Dom}(\mathcal{D}_0)$, we have $\mathcal{D}_0 F \in \text{Dom}(\mathcal{D}_1)$ and

$$(3.8) \quad \mathcal{D}_1 \mathcal{D}_0 F = 0.$$

This is because $\mathcal{D}_1 \mathcal{D}_0 F = 0$ for smooth F and the general result follows from the closedness of \mathcal{D}_0 and \mathcal{D}_1 as differential operators.

Proposition 3.1. *The associated Laplacian operator \square_φ is a densely-defined, closed, self-adjoint and non-negative operator on $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$.*

Proof. It is similar to the proof of proposition 4.2.3 of [5] for $\bar{\partial}$ -complex. We give the proof here for completeness.

As we mentioned before, \mathcal{D}_0 and \mathcal{D}_0^* as differential operators are both densely-defined and closed. \square_φ is densely-defined in the same way. For closedness of \square_φ , we need to show that for any $f_n \in \text{Dom}(\square_\varphi)$ such that $f_n \rightarrow f$ in $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ and $\square_\varphi f_n$ converges, we have $f \in \text{Dom}(\square_\varphi)$ and $\square_\varphi f_n \rightarrow \square_\varphi f$. Because $f_n \in \text{Dom}(\square_\varphi)$, we have

$$\begin{aligned} \langle \square_\varphi(f_n - f_m), f_n - f_m \rangle_\varphi &= \langle \mathcal{D}_0 \mathcal{D}_0^*(f_n - f_m), f_n - f_m \rangle_\varphi + \langle \mathcal{D}_1^* \mathcal{D}_1(f_n - f_m), f_n - f_m \rangle_\varphi \\ &= \|\mathcal{D}_0^*(f_n - f_m)\|_\varphi^2 + \|\mathcal{D}_1(f_n - f_m)\|_\varphi^2, \end{aligned}$$

and so $\mathcal{D}_0^* f_n$ and $\mathcal{D}_1 f_n$ converge in $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0)$ and $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$, respectively. It follows from the closedness of \mathcal{D}_0^* and \mathcal{D}_1 that $f \in \text{Dom}(\mathcal{D}_0^*) \cap \text{Dom}(\mathcal{D}_1)$ and

$$\mathcal{D}_0^* f_n \rightarrow \mathcal{D}_0^* f, \quad \mathcal{D}_1 f_n \rightarrow \mathcal{D}_1 f.$$

Note that $\mathcal{D}_0 \mathcal{D}_0^* f_n$ and $\mathcal{D}_1^* \mathcal{D}_1 f_n$ are orthogonal to each other by

$$\langle \mathcal{D}_0 \mathcal{D}_0^* f_n, \mathcal{D}_1^* \mathcal{D}_1 f_n \rangle_\varphi = \langle \mathcal{D}_1 \mathcal{D}_0 \mathcal{D}_0^* f_n, \mathcal{D}_1 f_n \rangle_\varphi = 0$$

by (3.8). So $\square_\varphi f_n = \mathcal{D}_0 \mathcal{D}_0^* f_n + \mathcal{D}_1^* \mathcal{D}_1 f_n$ converges implies that both $\mathcal{D}_0 \mathcal{D}_0^* f_n$ and $\mathcal{D}_1^* \mathcal{D}_1 f_n$ converge. It follows from the closedness of \mathcal{D}_0 and \mathcal{D}_1^* again that $\mathcal{D}_0^* f \in \text{Dom}(\mathcal{D}_0)$, $\mathcal{D}_1 f \in (\mathcal{D}_1^*)$ and

$$\mathcal{D}_0 \mathcal{D}_0^* f_n \rightarrow \mathcal{D}_0 \mathcal{D}_0^* f, \quad \mathcal{D}_1^* \mathcal{D}_1 f_n \rightarrow \mathcal{D}_1^* \mathcal{D}_1 f.$$

Therefore $f \in \text{Dom}(\square_\varphi)$ and $\square_\varphi f_n \rightarrow \square_\varphi f$. So \square_φ is a closed operator.

Define

$$(3.9) \quad L_1 := \mathcal{D}_0 \mathcal{D}_0^* + \mathcal{D}_1^* \mathcal{D}_1 + I \quad \text{on} \quad \text{Dom}(\square_\varphi).$$

It is sufficient to show that L_1^{-1} is self-adjoint. By a theorem of Von Neumann (cf. §1 in Chapter 8 in [18]), $(I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1}$ and $(1 + \mathcal{D}_1^* \mathcal{D}_1)^{-1}$ are automatically both bounded and self-adjoint, and so is

$$Q_1 = (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1} + (I + \mathcal{D}_1^* \mathcal{D}_1)^{-1} - I.$$

We claim that $Q_1 = L_1^{-1}$. Since

$$(1 + \mathcal{D}_0 \mathcal{D}_0^*)^{-1} - I = (I - (I + \mathcal{D}_0 \mathcal{D}_0^*)) (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1} = -\mathcal{D}_0 \mathcal{D}_0^* (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1},$$

we see that $\mathcal{R}(I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1} \subset \text{Dom}(\mathcal{D}_0 \mathcal{D}_0^*)$. Similarly, $\mathcal{R}(I + \mathcal{D}_1^* \mathcal{D}_1)^{-1} \subset \text{Dom}(\mathcal{D}_1^* \mathcal{D}_1)$, and so

$$(3.10) \quad Q_1 = (I + \mathcal{D}_1^* \mathcal{D}_1)^{-1} - \mathcal{D}_0 \mathcal{D}_0^* (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1}.$$

Since $\mathcal{D}_1 \mathcal{D}_0 = 0$ by (3.8), we have $\mathcal{R}(Q_1) \subset \text{Dom}(\mathcal{D}_1^* \mathcal{D}_1)$ and $\mathcal{D}_1^* \mathcal{D}_1 Q_1 = \mathcal{D}_1^* \mathcal{D}_1 (I + \mathcal{D}_1^* \mathcal{D}_1)^{-1}$. Similarly $\mathcal{R}(Q_1) \subset \text{Dom}(\mathcal{D}_0 \mathcal{D}_0^*)$ and $\mathcal{D}_0 \mathcal{D}_0^* Q_1 = \mathcal{D}_0 \mathcal{D}_0^* (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1}$. Consequently, $\mathcal{R}(Q_1) \subset \text{Dom}(L_1)$ and

$$L_1 Q_1 = \mathcal{D}_1^* \mathcal{D}_1 (I + \mathcal{D}_1^* \mathcal{D}_1)^{-1} + \mathcal{D}_0 \mathcal{D}_0^* (I + \mathcal{D}_0 \mathcal{D}_0^*)^{-1} + Q_1 = I$$

by (3.10). This together with the injectivity of L_1 implies that $L_1^{-1} = Q_1$. Thus L_1^{-1} is self-adjoint. So is its inverse L_1 (cf. §2 in Chapter 8 in [18] for this general property). \square

3.3. The canonical solution operator. *Proof of Theorem 1.1.* (1) The weighted L^2 -estimate (1.8) implies that

$$4 \|g\|_\varphi^2 \leq \|\mathcal{D}_0^* g\|_\varphi^2 + \|\mathcal{D}_1 g\|_\varphi^2 = (\square_\varphi g, g)_\varphi \leq \|\square_\varphi g\|_\varphi \|g\|_\varphi,$$

for $g \in \text{Dom}(\square_\varphi)$, by (3.7), i.e.

$$(3.11) \quad 4 \|g\|_\varphi \leq \|\square_\varphi g\|_\varphi.$$

Thus \square_φ is injective. This together with the self-adjointness of \square_φ by Proposition 3.1 implies the density of the range (cf. §2 in Chapter 8 in [18] for this general property). For fixed $f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$, the complex anti-linear functional

$$\lambda_f : \square_\varphi g \longrightarrow \langle f, g \rangle_\varphi$$

is then well-defined on a dense subset $\mathcal{R}(\square_\varphi)$ of $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$. It is finite since

$$|\lambda_f(\square_\varphi g)| = |\langle f, g \rangle_\varphi| \leq \|f\|_\varphi \|g\|_\varphi \leq \frac{1}{4} \|f\|_\varphi \|\square_\varphi g\|_\varphi$$

for any $g \in \text{Dom}(\square_\varphi)$, by (3.11). So λ_f can be uniquely extended a continuous anti-linear functional on $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$. By the Riesz representation theorem, there exists a unique element $h \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ such that $\lambda_f(F) = \langle h, F \rangle_\varphi$ for any $F \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$, and $\|h\|_\varphi = |\lambda_f| \leq \frac{1}{4} \|f\|_\varphi$. In particular, we have

$$\langle h, \square_\varphi g \rangle_\varphi = \langle f, g \rangle_\varphi$$

for any $g \in \text{Dom}(\square_\varphi)$. This implies that $h \in \text{Dom}(\square_\varphi^*)$ and $\square_\varphi^* h = f$, and so $h \in \text{Dom}(\square_\varphi)$ and $\square_\varphi h = f$ by self-adjointness of \square_φ . We write $h = N_\varphi f$. Then $\|N_\varphi f\|_\varphi \leq \frac{1}{4} \|f\|_\varphi$.

(2) Since $N_\varphi f \in \text{Dom}(\square_\varphi)$, we have $\mathcal{D}_0^* N_\varphi f \in \text{Dom}(\mathcal{D}_0)$, $\mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1^*)$, and

$$(3.12) \quad \mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f - \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f$$

by $\square_\varphi N_\varphi f = f$. Because f and $\mathcal{D}_0 F$ for any $F \in \text{Dom}(\mathcal{D}_0)$ are both \mathcal{D}_1 -closed, the above identity implies $\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f \in \text{Dom}(\mathcal{D}_1)$ and so $\mathcal{D}_1 \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f = 0$ by \mathcal{D}_1 acting in both sides. Then

$$0 = \langle \mathcal{D}_1 \mathcal{D}_1^* \mathcal{D}_1 N_\varphi f, \mathcal{D}_1 N_\varphi f \rangle_\varphi = \|\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f\|_\varphi^2,$$

i.e. $\mathcal{D}_1^* \mathcal{D}_1 N_\varphi f = 0$. Hence $\mathcal{D}_0 \mathcal{D}_0^* N_\varphi f = f$ by (3.12). Moreover, we have $\mathcal{D}_0^* N_\varphi f \perp A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ since $(F, \mathcal{D}_0^* N_\varphi f)_\varphi = (\mathcal{D}_0 F, N_\varphi f)_\varphi = 0$ for any $F \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. The estimate (1.7) follows from

$$\|\mathcal{D}_0^* N_\varphi f\|_\varphi^2 + \|\mathcal{D}_1 N_\varphi f\|_\varphi^2 = \langle \square_\varphi N_\varphi f, N_\varphi f \rangle_\varphi \leq \frac{1}{4} \|f\|_\varphi^2.$$

Corollary 3.1. *The weighted k -Bergman projection formula (1.9) holds.*

Proof. For $f \in \text{Dom}(\mathcal{D}_0)$, $\mathcal{D}_0 f$ is automatically \mathcal{D}_1 -closed. Apply Theorem 1.1 to $\mathcal{D}_0 f$ to get the canonical solution $\mathcal{D}_0^* N_\varphi \mathcal{D}_0 f$ orthogonal to $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. So $f - \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ by $\mathcal{D}_0(f - \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f) = 0$, and is exactly the projection of f to the weighted k -Bergman space. \square

Remark 3.1. *As in [21], we can use Theorem 1.1 to get compactly supported solution to the non-homogeneous k -Cauchy-Fueter equation (1.1) for \mathcal{D}_1 -closed $f \in C_0^1(\mathbb{R}^{4n}, \mathcal{V}_1)$, which implies Hartogs' phenomenon for k -regular functions.*

4. DECAY OF CANONICAL SOLUTIONS AND THE WEIGHTED k -BERGMAN KERNEL

4.1. The weighted k -Bergman projection and kernel. For $f \in L_\varphi^2(\Omega, \mathcal{V}_0)$, it has $k+1$ independent components $f_{0'0'} \dots 0'0', f_{1'0'} \dots 0'0', \dots, f_{1'1'} \dots 1'1'.$ We write

$$(4.1) \quad f = \begin{pmatrix} f_{0'0'} \dots 0'0' \\ f_{1'0'} \dots 0'0' \\ \vdots \\ f_{1'1'} \dots 1'1' \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_k \end{pmatrix},$$

where $f_j := f_{1' \dots 1' 0' \dots 0'}$ with j indices to be $1'$.

Note that for a sequence of k -regular functions $F_n \in L_\varphi^2(\Omega, \mathcal{V}_0)$ (i.e. $\mathcal{D}_0 F_n = 0$), if $F_n \rightarrow F$ in $L_\varphi^2(\Omega, \mathcal{V}_0)$, we have $\mathcal{D}_0 F = 0$ by the closedness of \mathcal{D}_0 . So $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ is a closed subspace of $L_\varphi^2(\Omega, \mathcal{V}_0)$. If $\{\psi_\alpha\}$ is an orthonormal basis of the space $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$, the weighted k -Bergman projection P can be write as $Pf = \sum_\alpha \langle f, \psi_\alpha \rangle_\varphi \psi_\alpha$.

Proposition 4.1. *If $f \in L_\varphi^2(\Omega, \odot^k \mathbb{C}^2)$ is k -regular, then each component of f is harmonic.*

Proof. It follows from

$$(4.2) \quad \overline{\mathcal{D}_0}^t \mathcal{D}_0 f = \begin{pmatrix} \Delta & 0 & \cdots & 0 & 0 \\ 0 & 2\Delta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2\Delta & 0 \\ 0 & 0 & \cdots & 0 & \Delta \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_k \end{pmatrix}$$

where $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_{4n}}^2$. See lemma 3.3 of [19] for this identity. \square

By Proposition 4.1, each component of a k -regular function is smooth. So for a fixed point $x \in \mathbb{R}^{4n}$, we can define complex linear functionals

$$l_j(f) = f_j(x)$$

for $f \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$, $j = 0, \dots, k$. Since f_j is harmonic by Proposition 4.1, we see that

$$(4.3) \quad |f_j(x)| = \left| \frac{1}{|B(x, 1)|} \int_{B(x, 1)} f_j(y) dV(y) \right| \leq \frac{1}{|B(x, 1)|} \|f\|_\varphi \left(\int_{B(x, 1)} e^{2\varphi(y)} dV(y) \right)^{\frac{1}{2}} \leq C_x \|f\|_\varphi,$$

where C_x only depends on x , not on f . Consequently, linear functionals l_j are bounded on $A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. By the Riesz representation theorem, there exists $K_j(\cdot, x) \in A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$ such that

$$f_j(x) = \langle f, K_j(\cdot, x) \rangle_\varphi = \sum_{l=0}^k \int_{\mathbb{R}^{4n}} f_l(y) \overline{K_{jl}(y, x)} e^{-2\varphi} dV.$$

It is obvious that $\langle g, K_j(\cdot, x) \rangle_\varphi = 0$ for any $g \perp A_{(k)}^2(\mathbb{R}^{4n}, \varphi)$. So $K(x, y) = \overline{K_{jl}(y, x)}$ is the kernel of the weighted k -Bergman projection P , which is a $(k+1) \times (k+1)$ matrix anti- k -regular in y . Then the integral formula (1.10) holds. Since an orthogonal projection P is self-adjoint on $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0)$, K has the Hermitian property $K(x, y) = \overline{K(y, x)}^t$, and so $K(x, y)$ is k -regular in x .

4.2. A localized a priori estimate and Caccioppoli-type estimate. It is known that the Caccioppoli-type estimate holds for many systems of PDEs of the divergence form by establishing localized a priori estimate of the following type.

Proposition 4.2. *There exists an absolute constant $C_0 > 0$ such that for any $f \in \text{Dom}(\square_\varphi)$ and real bounded Lipschitzian function η , we have estimates*

$$(4.4) \quad \begin{aligned} \|\eta \mathcal{D}_1 f\|_\varphi^2 + \|\eta \mathcal{D}_0^* f\|_\varphi^2 &\leq C_0 \left(\| |d\eta| \cdot f \|_\varphi^2 + | \langle \eta^2 f, \square_\varphi f \rangle_\varphi | \right), \\ \mathcal{E}_\varphi(\eta f, \eta f) &\leq C_0 \left(\| |d\eta| \cdot f \|_\varphi^2 + | \langle \eta^2 f, \square_\varphi f \rangle_\varphi | \right), \end{aligned}$$

where $|d\eta|^2 = \sum_{j=1}^{4n} \left| \frac{\partial \eta}{\partial x_j} \right|^2$.

Proof. Note that

$$\delta_{A'_1}^A (\eta f_{A'_2 \dots A'_k A}) = \eta \delta_{A'_1}^A f_{A'_2 \dots A'_k A} + Z_{A'_1}^A \eta \cdot f_{A'_2 \dots A'_k A}$$

by $\delta_{A'_1}^A = Z_{A'_1}^A - 2Z_{A'_1}^A \varphi$ in (2.9). Then taking summation over A and symmetrising $(A'_1 \dots A'_k)$, we get

$$(4.5) \quad [\mathcal{D}_0^*(\eta f)]_{A'_1 \dots A'_k} = \eta [\mathcal{D}_0^*(f)]_{A'_1 \dots A'_k} + \sum_{A=0}^{2n-1} Z_{(A'_1}^A \eta \cdot f_{A'_2 \dots A'_k A)}.$$

On the other hand, for fixed $A'_1 \dots A'_k$, we have

$$(4.6) \quad \begin{aligned} \left| \sum_A Z_{(A'_1}^A \eta \cdot f_{A'_2 \dots A'_k A)} \right| &= \frac{1}{k} \left| \sum_{s=1}^k \sum_A Z_{A'_s}^A \eta \cdot f_{\dots A'_1 \dots A'_k A} \right| \\ &\leq \frac{1}{k} \sum_{s=1}^k \left(\sum_A |Z_{A'_s}^A \eta|^2 \right)^{\frac{1}{2}} \left(\sum_A |f_{A'_1 \dots \widehat{A'_s} \dots A'_k A}|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

by using (3.5) and Cauchy-Schwarz inequality and f symmetric in the primed indices. Note that it directly follows from definition (2.1) of $Z_{AA'}$'s that

$$\sum_{A=0}^{2n-1} |Z_{AA'} \eta|^2 = |d\eta|^2$$

for fixed $A' = 0'$ or $1'$. Then by raising indices, we get

$$\sum_{A=0}^{2n-1} |Z_{0'}^A \eta|^2 = \sum_{A=0}^{2n-1} |\overline{Z_A^{0'}} \eta|^2 = \sum_{A=0}^{2n-1} |Z_{A1'} \eta|^2 = |d\eta|^2,$$

and so is the sum of $|Z_1^A \eta|^2$. Apply these to (4.6) to get

$$(4.7) \quad \sum_{A'_1, \dots, A'_k} \left\| \sum_A Z_{(A'_1 \eta \cdot f_{A'_2 \dots A'_k})A}^A \right\|_\varphi^2 \leq 2 \| |d\eta| \cdot f \|_\varphi^2.$$

Thus we get the estimate

$$\| \mathcal{D}_0^*(\eta f) \|_\varphi^2 \leq \| \eta \mathcal{D}_0(f) \|_\varphi^2 + 2 \| |d\eta| \cdot f \|_\varphi^2,$$

by (4.5), and simultaneously,

$$(4.8) \quad \| \eta \mathcal{D}_0^* f \|_\varphi^2 \leq \| \mathcal{D}_0^*(\eta f) \|_\varphi^2 + 2 \| |d\eta| \cdot f \|_\varphi^2.$$

Note that by (4.5) again, we get

$$\begin{aligned} \| \mathcal{D}_0^*(\eta f) \|_\varphi^2 &= \sum_{A'_1 \dots A'_k} \left(\mathcal{D}_0^*(\eta f)_{A'_1 \dots A'_k}, \sum_A Z_{(A'_1 \eta \cdot f_{A'_2 \dots A'_k})A}^A + \eta(\mathcal{D}_0^* f)_{A'_1 \dots A'_k} \right)_\varphi \\ &= \sum_{A'_1 \dots A'_k} \left(\mathcal{D}_0^*(\eta f)_{A'_1 \dots A'_k}, \sum_A Z_{(A'_1 \eta \cdot f_{A'_2 \dots A'_k})A}^A \right)_\varphi + \langle \mathcal{D}_0^*(\eta f), \eta \mathcal{D}_0^* f \rangle_\varphi \\ &\leq \kappa \| \mathcal{D}_0^*(\eta f) \|_\varphi^2 + \frac{1}{\kappa} \| |d\eta| \cdot f \|_\varphi^2 + \langle \eta f, \mathcal{D}_0(\eta \mathcal{D}_0^* f) \rangle_\varphi \end{aligned}$$

by using estimates (4.6)-(4.7) and the trivial inequality $2|ab| \leq \kappa|a|^2 + \frac{1}{\kappa}|b|^2$ for any $\kappa > 0$. Thus if we choose $\kappa = 1/2$, we get

$$(4.9) \quad \| \mathcal{D}_0^*(\eta f) \|_\varphi^2 \leq 4 \| |d\eta| \cdot f \|_\varphi^2 + 2 \langle \eta f, \mathcal{D}_0(\eta \mathcal{D}_0^* f) \rangle_\varphi.$$

But

$$\begin{aligned} |\langle \eta f, \mathcal{D}_0(\eta \mathcal{D}_0^* f) \rangle_\varphi| &\leq |\langle \eta f, \eta \mathcal{D}_0 \mathcal{D}_0^* f \rangle_\varphi| + \sum_{A, A'_2, \dots, A'_k} \left| \left(\eta f_{A'_2 \dots A'_k A}, \sum_{A'_1} Z_{A'_1}^{A'_1} \eta \cdot (\mathcal{D}_0^* f)_{A'_1 \dots A'_k} \right)_\varphi \right| \\ (4.10) \quad &\leq |\langle \eta^2 f, \mathcal{D}_0 \mathcal{D}_0^* f \rangle_\varphi| + \sum_{A'_1, \dots, A'_k} \sum_A \left| \left(\overline{Z_A^{A'_1}} \eta f_{A'_2 \dots A'_k A}, \eta (\mathcal{D}_0^* f)_{A'_1 \dots A'_k} \right)_\varphi \right| \\ &\leq |\langle \eta^2 f, \mathcal{D}_0 \mathcal{D}_0^* f \rangle_\varphi| + \frac{1}{\kappa} \| |d\eta| \cdot f \|_\varphi^2 + \kappa \| \eta \mathcal{D}_0^* f \|_\varphi^2 \end{aligned}$$

by applying estimates similar to (4.6)-(4.7) in the third inequality. Now Substitute (4.10) to (4.9) and using (4.8) to control the term $\kappa \| \eta \mathcal{D}_0^* f \|_\varphi^2$, we find that there exists a constant $C_0 > 0$ such that

$$\| \mathcal{D}_0^*(\eta f) \|_\varphi^2 \leq C_0 \left(\| |d\eta| \cdot f \|_\varphi^2 + |\langle \eta^2 f, \mathcal{D}_0 \mathcal{D}_0^* f \rangle_\varphi| \right).$$

Similarly,

$$\mathcal{D}_1(\eta f)_{ABA'_2 \dots A'_k} = \eta(\mathcal{D}_1 f)_{ABA'_2 \dots A'_k} + 2 \sum_{A'_1=0', 1'} Z_{[A}^{A'_1} \eta \cdot f_{B]A'_1 \dots A'_k}$$

by definition, and so

$$\begin{aligned} \| \mathcal{D}_1(\eta f) \|_\varphi^2 &\leq \| \eta \mathcal{D}_1(f) \|_\varphi^2 + 4n \| |d\eta| \cdot f \|_\varphi^2, \\ \| \eta \mathcal{D}_1 f \|_\varphi^2 &\leq C_0 \left(\| |d\eta| \cdot f \|_\varphi^2 + |\langle \eta^2 f, \mathcal{D}_1^* \mathcal{D}_1 f \rangle_\varphi| \right). \end{aligned}$$

The result follows. \square

As a corollary, we get Caccioppoli-type estimate.

Proposition 4.3. *Suppose that $\varphi(x) = |x|^2$. If $\square_\varphi F = 0$ on $B(x, R) \subset \mathbb{R}^{4n}$, then for $r < R$, we have*

$$\int_{B(x, r)} |\mathcal{D}_0^* F|^2 e^{-2\varphi} dV \leq \frac{C}{(R-r)^2} \int_{B(x, R)} |F|^2 e^{-2\varphi} dV$$

for some constant C only depending on n, k, R and r .

Proof. Let η be a $C_0^\infty(B(x, R))$ function such that $\eta \equiv 1$ on $B(x, r)$. By the localized a priori estimate (4.4) in Proposition 4.2, we get

$$\|\chi_{B(x, r)} \mathcal{D}_0^* F\|_\varphi^2 \leq \|\eta \mathcal{D}_0^* F\|_\varphi^2 \leq C_0 \left(\|d\eta\| \cdot \|F\|_\varphi^2 + |\langle \eta^2 F, \square_\varphi F \rangle_\varphi| \right) = C_0 \|d\eta\|_\infty^2 \|\chi_{B(x, R)} \cdot F\|_\varphi^2$$

since $\square_\varphi F = 0$ on $\text{supp } \eta$ and $d\eta$ is supported in $B(x, R)$. The result follows by choosing η . \square

4.3. Decay of canonical solutions and the weighted k -Bergman kernel.

Theorem 4.1. *Suppose that $\varphi(x) = |x|^2$, $k = 2, 3, \dots$, and that $f \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ is compactly supported in $B(y, r_0)$. Then the canonical solution $u = \mathcal{D}_0^* N_\varphi f$ has the following pointwise estimate: there exists $\varepsilon > 0$ only depending on r_0 and constant $C > 0$ only depending on n, k and ε such that*

$$(4.11) \quad |u(x)| \leq C e^{|x|^2 + \frac{\varepsilon}{2}|x| - \varepsilon|x-y|} \|f\|_\varphi$$

for any x such that $|x - y| > r_0 + 2$.

Proof. For the canonical solution $u = \mathcal{D}_0^* N_\varphi f$, we have $\mathcal{D}_0 u = f$ vanishing outside of $B(y, r_0)$. Consequently, each component of u is harmonic outside of $B(y, r_0)$ by Proposition 4.1. By the mean value formula for harmonic functions, we get

$$(4.12) \quad \begin{aligned} |u(x)|^2 &= \left| \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} u(x') dV \right|^2 \\ &\leq \frac{1}{|B(x, \delta)|^2} \int_{B(x, \delta)} |u(x')|^2 e^{-2|x'|^2} dV(x') \cdot \int_{B(x, \delta)} e^{2|x'|^2} dV(x') \\ &\leq C'_\delta e^{2|x|^2 + 4\delta|x|} \int_{B(x, 1)} |N_\varphi f(x')|^2 e^{-2|x'|^2} dV(x') \end{aligned}$$

for some constant $C'_\delta > 0$ only depending on $n, \delta < 1$ and any x such that $|x - y| > r_0 + 1$. Here in the last inequality we apply Caccioppoli-type estimate in Proposition 4.3 to $F = N_\varphi f$ with $\square_\varphi N_\varphi f = f = 0$ outside of $B(y, r_0)$, and $e^{|x'|^2} \leq e^{|x|^2 + 2\delta|x| + \delta^2}$ for $x' \in B(x, \delta)$. We choose $\delta = \frac{\varepsilon}{4}$ for ε determined later.

For fixed x outside of $B(y, r_0)$, consider the Lipschitzian function

$$b(x') := \min\{|x' - y|, |x - y|\}.$$

Let $l : [0, \infty) \rightarrow [0, 1]$ be the Lipschitzian function vanishing on $[0, r_0]$, equal to 1 on $[r_0 + 1, \infty)$, and affine in between. Set $\eta(x') = l(|x' - y|)$. Applying weighted L^2 estimate (1.8) and the localized a priori estimate in Proposition 4.2 to $N_\varphi f$ with η replaced by $\eta e^{\varepsilon b}$, we get

$$\begin{aligned} \int_{\mathbb{R}^{4n}} |\eta e^{\varepsilon b} N_\varphi f(x')|^2 e^{-2\varphi} dV(x') &\leq \mathcal{E}_\varphi(\eta e^{\varepsilon b} N_\varphi f, \eta e^{\varepsilon b} N_\varphi f) \\ &\leq C_0 \|d(\eta e^{\varepsilon b})\| \cdot \|N_\varphi f\|_\varphi^2 + C_0 (\eta^2 e^{2\varepsilon b} N_\varphi f, \square_\varphi N_\varphi f)_\varphi \\ &\leq C_0 \int_{\mathbb{R}^{4n}} (|d\eta| e^{\varepsilon b} N_\varphi f(x')|^2 + (4n\varepsilon)^2 |\eta e^{\varepsilon b} N_\varphi f(x')|^2) e^{-2\varphi} dV(x') \end{aligned}$$

since the Lipschitzian constant of b is 1 and $\square_\varphi N_\varphi f = f = 0$ on the support of η ($= B(y, r_0)^c$). Hence if we choose ε sufficiently small (e.g. $C_0(4n\varepsilon)^2 \leq \frac{1}{2}$), we get

$$\begin{aligned} \int_{\mathbb{R}^{4n}} |\eta e^{\varepsilon b} N_\varphi f(x')|^2 e^{-2\varphi} dV(x') &\leq 2C_0 \int_{\mathbb{R}^{4n}} ||d\eta| e^{\varepsilon b} N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \\ &\leq C'' \int_{B(y, r_0+1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \end{aligned}$$

for some constant $C'' > 0$, by $d\eta$ supported in $B(y, r_0 + 1)$ and b uniformly bounded on $B(y, r_0 + 1)$ ($|b(x')| < r_0 + 1$). But $b(x') \geq |x - y| - 1$ for $x' \in B(x, 1)$, and so the above estimate implies that

$$\int_{B(x, 1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x') \leq C'' e^{-2\varepsilon(|y-x|-1)} \int_{B(y, r_0+1)} |N_\varphi f(x')|^2 e^{-2\varphi} dV(x').$$

Substituting this into (4.12), we get the result by the boundedness of N_φ on $L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_1)$ by Theorem 1.1 (1). \square

Proof of Theorem 1.3. For fixed $y \in \mathbb{R}^{4n}$, let η_y be a smooth radial function supported in the ball $B(y, \delta)$ ($\delta < 1$) such that $\int \eta_y(y') dV(y') = 1$. Set

$$(4.13) \quad f_y(y') = \begin{pmatrix} \vdots \\ 0 \\ \eta_y(y') e^{2|y'|^2} \\ 0 \\ \vdots \end{pmatrix} \in L_\varphi^2(\mathbb{R}^{4n}, \mathcal{V}_0)$$

for fixed j , where only j -th entry is nonvanishing. Note that

$$Pf_y(x) = \int_{\mathbb{R}^{4n}} K(x, y') f_y(y') e^{-2|y'|^2} dV(y') = \int_{\mathbb{R}^{4n}} K(x, y') \begin{pmatrix} \vdots \\ 0 \\ \eta_y(y') \\ 0 \\ \vdots \end{pmatrix} dV(y') = \begin{pmatrix} K(x, y)_{0j} \\ \vdots \\ K(x, y)_{kj} \end{pmatrix}$$

by applying the mean value formula for harmonic functions to each component of $K(x, \cdot)$, since $\eta_y(\cdot)$ is constant on each sphere centered at y . Hence the j -th column of $(k+1) \times (k+1)$ -matrix K is

$$\begin{pmatrix} K(x, y)_{0j} \\ \vdots \\ K(x, y)_{kj} \end{pmatrix} = Pf_y(x) = f_y(x) - (\mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_y)(x),$$

by the identity (1.9). The exponential decay of the canonical solution in Theorem 4.1 implies that there exists a constant $C > 0$ only depending on ε, n, k such that

$$|(\mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_y)(x)| \leq C e^{|x|^2 + \frac{\varepsilon}{2}|x| - \varepsilon|x-y|} \|\mathcal{D}_0 f_y\|_\varphi$$

for any x such that $|x-y| > 3$, since $\mathcal{D}_0 \mathcal{D}_0^* N_\varphi \mathcal{D}_0 f_y = \mathcal{D}_0 f_y$ is supported in $B(y, 1)$. Note that $|\mathcal{D}_0 f_y(y')| \leq C_3 e^{2|y'|^2} (|y'| + 1) \chi_{B(y, \delta)}$ for some constant $C_3 > 0$ depending on n, δ , by direct differentiation (4.13). It is direct to check that $\|\mathcal{D}_0 f_y\|_\varphi \leq C_4 e^{|y|^2 + 5\delta|y|}$ for some constant $C_4 > 0$ depending on n, δ . The result follows by choose small δ .

Remark 4.1. Our estimate (1.11) has an extra factor $e^{\frac{\varepsilon}{2}(|x|+|y|)}$ compared to the estimate

$$|K(x, y)| \leq Ce^{|x|^2+|y|^2-\varepsilon|x-y|},$$

for the Bergmann kernel in complex analysis. But when $|y|$ is large compared to $|x|$, e.g. $|y| \geq 4|x|$,

$$|K(x, y)| \leq Ce^{|x|^2+|y|^2-\frac{\varepsilon}{8}|y|},$$

which has similar exponential decay with respect to the measure $e^{-|y|^2}dV$ as in the complex case.

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